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LINE TRANSVERSALS FOR HOMOTHETICAL SYSTEMS OF POLYGONS IN $\mathbb{R}^2$

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Abstract. The problem considered in this article is: 'for given finite system of convex polygons in the plane which have no transversal, find such homothety transformations of polygons (having fixed centres inside given polygons) with minimal similarity ratio $c > 1$ that the transformed system has a transversal'. We prove that in this 'minimal configuration', we can always find three polygons (two of them lying in distinct halfplanes determined by the transversal), for which the transversal is also the tangent line.

1. Introduction, notation

Definition 1.1. The **line transversal** to a system of convex sets in $\mathbb{R}^d$, $d \geq 2$ is the line, which intersects every member of the system.

For brief overview of recent results in geometric transversal theory, see [1]. In this article we restrict our attention to the convex polygons in the plane.

Definition 1.2. We will call the finite system $\mathcal{S} = \{P_1, P_2, \dots, P_n\}$ of convex polygons in $\mathbb{R}^2$ the **initial configuration** if no transversal of $\mathcal{S}$ exists.

For example, the smaller, filled polygons on Fig. 1 form an initial configuration.

In this paper we will use the notation $\mathcal{H}_{S,c}$ for the homothety transformation with center S and similarity ratio c given by:

$$\overrightarrow{\mathcal{H}_{S,c}(X) - S} = c\,\overrightarrow{(X - S)}, \text{ or } \mathcal{H}_{S,c}(X) = S + c\,\overrightarrow{(X - S)}, \; X \in \mathbb{R}^2.$$

Let P be the convex polygon and $S \in P$ an interior point of P. Let us denote by

$$\mathcal{H}_{S,c}(P) = \{\mathcal{H}_{S,c}(X), \, X \in P\}$$

the image of polygon P in transformation $\mathcal{H}_{S,c}$. Throughout this paper we will fix the center S of homothety transformation applied to polygon P to be the centroid of P, i.e. the point $T = (x_T, y_T)$, with coordinates x_T, resp. y_T determined by the arithmetic means of x–coordinates (resp. y–coordinates) of vertices of P. In this case we will use the simple notation $\mathcal{H}_c(P)$ for the image of P.

We use the notation $\mathcal{S}_c$ for the image of finite system $\mathcal{S} = \{P_1, P_2, \dots, P_n\}$ of polygons, obtained by applying the transformations $\mathcal{H}_c$ to every polygon from $\mathcal{S}$.

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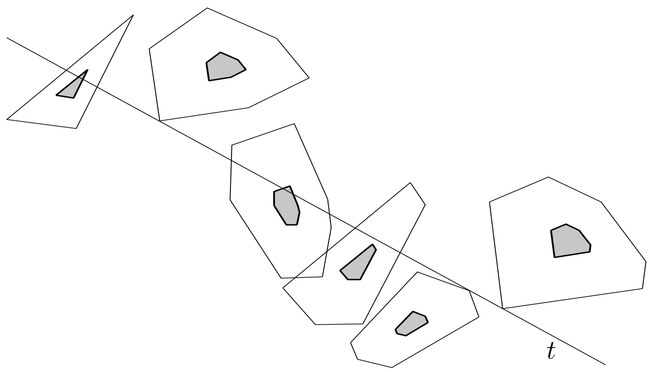

Figure 1. Initial configuration and its minimal configuration.

For $c > 1$ (resp. $c < 1$) the resulting image $\mathcal{S}_c$ can be described as the configuration consisting of polygons of $\mathcal{S}$ expanded (resp. shrinked) around their centroids as can be seen on Fig. 1 (bigger polygons drawn by thin lines).

It is clear that every finite system $\mathcal{S}$ of convex polygons in the plane can be transformed to an initial configuration $\mathcal{S}_c$ by taking the similarity ratio c sufficiently small, except for the degenerate case, where all the centroids of polygons of $\mathcal{S}$ are collinear.

For given initial configuration $\mathcal{S}$ there exists a real number $c_m > 1$ such that for every positive $c < c_m$ the configuration S_c has no line traversal and the configuration S_{c_m} has a line traversal.

Definition 1.3. We will call the number c_m the **minimal expansion ratio** and the configuration $\mathcal{S}_{c_m}$ the **minimal configuration** for the given initial configuration $\mathcal{S}$.

Example of minimal configuration (big non-filled polygons) for initial configuration (given by small filled polygons) can be seen on Fig. 1.

2. Main result

We can see from Fig. 1 that, in minimal configuration, there are three polygons, for which the (unique) transversal t is also the tangent line. This is the general property of minimal configuration, as we state in the next theorem.

Theorem 2.1. *Let $\mathcal{S}$ be the initial configuration. The configuration $\mathcal{S}_{c_m}$ with $c_m > 1$ is the minimal configuration for $\mathcal{S}$ if and only if there are three polygons $P_i, P_j, P_k \in \mathcal{S}_{c_m}$, which intersect the transversal t to $\mathcal{S}_{c_m}$ either in one point (vertex) or have one common side with t. Polygons P_i, P_j, P_k cannot lie in the same halfplane determined by transversal t. The transversal t for minimal configuration is unique.*

Before the proof of theorem 2.1 let us make some remarks about concepts, involved in the proof.

Definition 2.2. Let us have the line q and the convex polygon P. Then there exists a nonnegative real number c, such that the line q is the tangent line of transformed polygon $\mathcal{H}_c(P)$. We will call such c the **correcting factor of polygon P with respect to line q**. If the centroid of P lies on the line q, we put $c = 0$.

We will use the above definition mainly in the case, when the line q has nonempty intersection with interior of the polygon P. In such case, we will speak also about **shrinking factor**, see Fig. 2. For the line q_1 the shrinking factor is $c_1 = 3/4$ and for q_2 the shrinking factor is $c_2 = 1/2$.

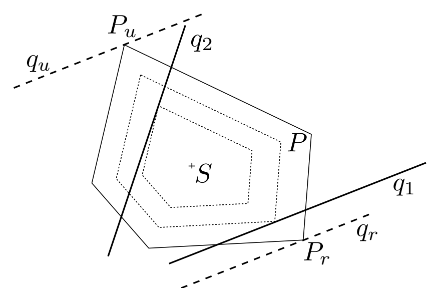

Figure 2. Shrinking factor.

It is not difficult to express the correcting factor explicitly. For given line q and polygon P we can construct the 'minimal strip' containing P with boundary lines, which are the translates of q (see Fig 2, for the line q_1 the strip is bounded by the lines q_u, q_r). Let us denote by $\varrho(A, p)$ the distance of point A from the line p and by q_s the line, parallel with q such that $S \in q$ then the correcting factor can be expressed as:

$$c = \begin{cases} \dfrac{\varrho(S,q)}{\varrho(S,q_u)} & \text{if } q, q_u \text{ lie in the same halfplane, determined by } q_s \\ \dfrac{\varrho(S,q)}{\varrho(S,q_r)} & \text{if } q, q_r \text{ lie in the same halfplane, determined by } q_s. \end{cases}$$

Thus, the following proposition holds.

Proposition 2.3. *Let us take $t = \varrho(S, q)$ as independent variable. Then the function $c = c(t)$, $t \in \mathbb{R}$ is continuous, piecewise linear, V-shaped function.*

Proof of Theorem 2.1. Let us consider the initial configuration $\mathcal{S}$. We will show that the configuration $\mathcal{S}_c$, in which less than three polygons have the transversal as tangent line, cannot be minimal.

Case 1. If, for some configuration $\mathcal{S}_c = \{P_1, P_2, \ldots, P_n\}$, there exists a transversal t, which has nonempty intersections with interiors of all polygons from $\mathcal{S}_c$, then all shrinking factors $c_1, c_2, \ldots, c_n$ have values less than 1, and so has also the number

$$c_m = \max\{c_1, c_2, \ldots, c_n\} < 1 \text{ and we have } \tilde{c} = c\, c_m < c.$$

Then, the configuration $\mathcal{S}_{\tilde{c}}$ has the same transversal t, but smaller similarity ratio, so the configuration $\mathcal{S}_c$ cannot be minimal.

Case 2. Now let us consider the second case, when for some configuration $\mathcal{S}_c$ there is exactly one polygon $P_k \in \mathcal{S}_c$ with no interior points belonging to given transversal t of $\mathcal{S}_c$ (i.e. t is the tangent line of P_k). Let us introduce the function $\varphi(c)$ defined by formula:

$$\varphi(c) = c - \max\{c_1(c), c_2(c), \ldots, c_{k-1}(c), c_{k+1}(c), \ldots, c_n(c)\},$$

where $c_j(c), j \neq k$ are the correcting factors of polygons $P_j \neq P_k$, when the transversal t is translated so that the shrinking factor of polygon P_k will become c (see Fig. 3).

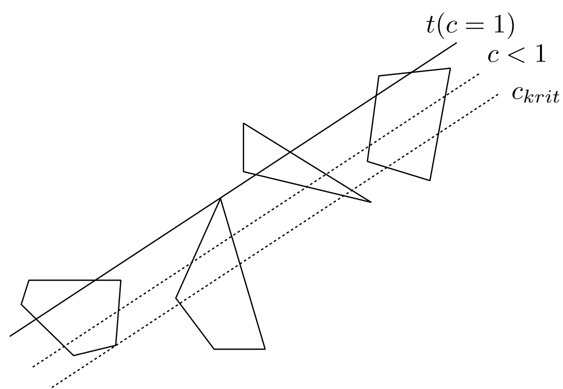

Figure 3. Case of one polygon touching transversal.

For $c = c_k = 1$ we have the untranslated transversal t. We will consider only values of c in interval $c_{krit} \leq c \leq 1$, where c_{krit} is such number that for $c < c_{krit}$ the translated line t remains no longer the transversal of $\mathcal{S}_c$, so $c_j = 1$ for some $j \neq k$ if $c = c_{krit}$ (cf. Fig. 3). It is clear that $c_{krit} < 1$, because if it is not true, then some other polygon $P_i, i \neq k$ should touch the transversal t. It can be easily shown that the function $\varphi(c)$ is continuous (it follows from proposition 2.3). Now, we have:

$$\varphi(1) = 1 - \max\{c_1, c_2, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n\} > 0, \quad \varphi(c_{krit}) = c_{krit} - 1 < 0$$

it follows that for some $\tilde{c}$, $c_{krit} < \tilde{c} < 1$, $\varphi(\tilde{c}) = 0$ or

$$\tilde{c} = c_k = \max\{c_1(\tilde{c}), c_2(\tilde{c}), \ldots, c_{k-1}(\tilde{c}), c_{k+1}(\tilde{c}), \ldots, c_n(\tilde{c})\}.$$

For such $\tilde{c}$ the translated transversal is the also transversal for configuration $\mathcal{S}_{c\,\tilde{c}}$, so the original configuration $\mathcal{S}_c$ cannot be minimal.

Case 3. Finally, lets us consider some configuration $\mathcal{S}_c$ in which there are exactly two polygons $P_i, P_j \in \mathcal{S}_c$ with no interior points belonging to given transversal t of $\mathcal{S}_c$. It is sufficient to take into account only such configuration, where P_i, P_j lie in distinct halfplanes, determined by the transversal t, because if they are situated

in the same halfplane, we can proceed as in preceding case. Now, consider the function:

$$\varphi(c) = c - \max\{c_k(c);\ k = 1, 2, \ldots, n,\ k \neq i, k \neq j\},$$

where $c_k(c)$ are the correcting factors of polygons P_k, when the transversal t is moved so that the shrinking factor of both polygons P_i, P_j will become c (Fig. 4).

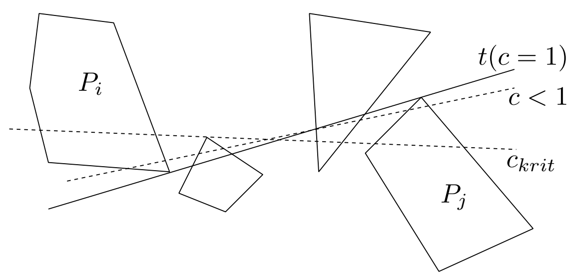

Figure 4. Case of two polygon touching transversal.

Analogically as in case 2. we will consider only values of c lying in interval $c_{krit} \leq c \leq 1$, where c_{krit} is such number that for $c < c_{krit}$ the moved line t remains no longer the transversal of $\mathcal{S}_c$, so $c_l(c_{krit}) = 1$ for some $l \neq i, l \neq j$.

If the function $\varphi(c)$ is continuous, we can proceed exactly as in case 2., showing that moved transversal t is also the transversal for some configuration $\mathcal{S}_{\hat{c}}$ with $\hat{c} < c$; thus, $\mathcal{S}_c$ cannot be the minimal configuration.

It is sufficient to ascertain the continuity of individual coefficients $c_k(c)$. Let us look at Fig. 5.

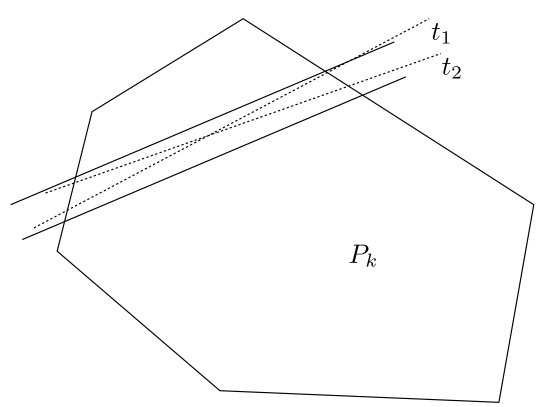

Figure 5. Two lines t_1, t_2 with slightly different values of c.

When we consider two moved transversals t_1, t_2 for slightly different values c_1, c_2 of c, it is clear, that the angle between them can be made arbitrarily small

by choosing the difference $|c_1 - c_2|$ to be sufficiently small. Now, we can enclose both $t_1 \cap P_k$ and $t_2 \cap P_k$ in a strip, bounded by parallel lines. Because the width of the strip can be made also arbitrarily small by choosing c_1, c_2 to be very close, we have reduced the problem of continuity of $c_k(c)$ to the analogical problem for translates, which was analysed in case 2.

So, considering above three cases, we prove that if $\mathcal{S}_c$ is minimal configuration for $\mathcal{S}$ then (at least) three polygons from $\mathcal{S}_c$ exist, which have the transversal t to $\mathcal{S}_c$ as the tangent line. All three polygons cannot lie in the same halfplane, determined by t, because such configuration can be easily reduced by method used in case 2.

Conversely, if for some configuration $\mathcal{S}_c$ we can find three polygons from $\mathcal{S}_c$, which have the transversal t as common tangent and these polygons do not lie in the same halfplane, determined by t, then it is obvious (see Fig. 6; only three mentioned polygons are shown, the location of other polygons from $\mathcal{S}_c$ does not matter) that $\mathcal{S}_c$ is minimal configuration.

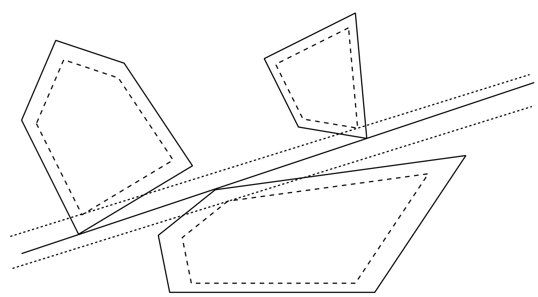

Figure 6. Configuration with three polygons touching transversal.

Really, by any change of c to smaller value $\tilde{c}$, we obtain a strip of nonzero width, bounded by parallel lines, which separates the mentioned polygons, so we cannot find the line (transversal), which has common points with all three polygons. From this fact it is also clear that the traversal for minimal configuration is unique. □

References

[1] Goodman D. E., Pollack R. and Wenger R., *Geometric transversal theory*, Algorithms and Combinatorics, vol. 10, Springer-Verlag, Heidelberg, 1993.

[2] Kaukič M., *Dynamical properties of finite systems of intervals in* $\mathbf{R}^2$, Journal of Information, Control and Management Systems, vol. 2, No. 2, 2004, 147-152

[3] Kaukič M., *On solution of error-adaptation problem*, Proceedings of 4-th International Conference Aplimat 2005, 299-304

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